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LETTER TO THE EDITOR

A boson–quasiboson mapping and Dirac quantization

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Abstract. In this letter we derive a boson–quasiboson mapping, patterned on the example of boson–quasifermion mappings. We shall show that this mapping, derived by purely algebraic means, is equivalent to the use of Dirac quantization. This may shed some light on the meaning of such mappings.

In this letter we discuss the realization of a boson–quasiboson mapping, in analogy with boson–quasifermion mappings [1]. The original impetus for the latter was the quantum Bogoliubov–Valatin transformation [2] as well as work on the vector coherent state method [3], both of which can be shown to be closely linked [1]. The basic idea in these mappings is to introduce a new degree of freedom that describes a 'collective' excitation. With this extra degree of freedom, we have now succeeded in obtaining an over-complete set of degrees of freedom. All applications to date concern systems of fermions, and this seems to hide the simple interpretation of these mappings. One might attempt to formulate the methods in terms of the constrained quantization of Grassmanian canonical variables, but this will hide the simplicity of the underlying methodology. If we make a similar mapping for a bosonic system no such problems arise, however.

In this letter we discuss a mapping of quantum mechanical operators to non-redundant degrees of freedom for a simple model containing two kinds of bosons. This is then compared to the Dirac quantization of the underlying classical Hamiltonian. As one might hope, the methods agree. Finally we mention where such boson–boson mappings might be useful.

For the purpose of this letter it is sufficient to consider the two-level harmonic oscillator

$$H=\omega_+a_+^\dagger a_+ +\omega_- a_-^\dagger a_- . \tag{1}$$

The mapping is most easily introduced as a mapping of the Fock-space states, which in its turn induces a mapping of operators. We rewrite a state of the form $|n_+,n_-\rangle=(a_+^\dagger)^{n_+}(a_-^\dagger)^{n_-}|0\rangle$ by introducing a new quantum number, N, for the number of pairs

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$a_+^\dagger a_-^\dagger$. We then introduce a boson $B^\dagger$ that describes the combined action of one such pair $a_+^\dagger a_-^\dagger$. We thus have a mapping

$$|n_+, n_-\rangle \to |Nn\sigma) \tag{2}$$

where

$$N = \min(n_+, n_-)$$
$$n = |n_+ - n_-|$$
$$\sigma = \begin{cases} -1 & n_+ < n_- \\ 0 & n_+ = n_- \\ 1 & n_+ > n_- . \end{cases} \tag{3}$$

Since the boson $B^\dagger$ replaces $a_+^\dagger a_-^\dagger$ it is reasonable to assume that the proportionality between the $B^\dagger$ and $a_+^\dagger a_-^\dagger$ is a simple function of the number operators, ($n_\pm = a_\pm^\dagger a_\pm$)

$$B^\dagger = x(\hat{n}_+, \hat{n}_-) a_+^\dagger a_-^\dagger . \tag{4}$$

The boson commutation relation $[B, B^\dagger] = 1$ now leads to the condition

$$x(n_+ + 1, n_- + 1)^2 (n_+ + 1)(n_- + 1) - x(n_+, n_-)^2 n_+ n_- = 1. \tag{5}$$

Through induction we can easily find a solution of this equation

$$x(n_+, n_-)^2 = \frac{1}{\max(n_+, n_-)}. \tag{6}$$

Clearly we find that $[B^\dagger, a_\pm] \neq 0$, which shows that we have an over-complete set of degrees of freedom. We would like to introduce, in analogy with the boson–quasifermion mappings discussed in Klein and Marshalek [1], a boson–quasiboson mapping that gets rid of this ambiguity. We thus introduce a set of quasiboson operators $b_\pm^\dagger$ that resemble $a_\pm^\dagger$ as closely as possible, but describe only the remaining degrees of freedom, i.e. they cannot form the boson $B^\dagger$. This implies the constraint $b_+^\dagger b_-^\dagger = 0$. As will be shown later, it is then no longer true that $[b_+^\dagger, b_+] = 1$ and $[b_+^\dagger, b_-] = 0$, but we can and will impose the condition $[B^\dagger, b_\pm] = 0$.

The way to proceed is as follows: we map both the algebra spanned by the bilinear operators $a_\alpha^\dagger a_\beta$, as well as the single operators $a^\dagger$. The trick is to find all non-trivial tensors in the new space that transform the same as in the old space, and multiply those with functions of the number operators in the new space. We then require equality between commutation relations of mapped operators with those of the original operators. It is known from the fermion case that one can assume that the algebra of linear with bilinear operators remains the same:

$$[b_\alpha^\dagger b_\alpha, b_\beta^\dagger] = \delta_{\alpha\beta} b_\beta^\dagger . \tag{7}$$

The states $|Nn\sigma)$ are now generated by the new operators as

$$|Nn\sigma) \propto B^{\dagger N} (b_\sigma^\dagger)^n |0) \tag{8}$$

where $|0)$ denotes the vacuum for the operators $b_\pm, B$.

The mapping of $a_\pm^\dagger$ (denoted $\Gamma(a_\pm^\dagger)$) is given by a some of all possible tensors that have the same action as $a^\dagger$ (we use $\Gamma(.)$ to denote the map of an operator, following [3])

$$\Gamma(a_\pm^\dagger) = f_\pm(\hat{N}, \hat{\mathcal{N}}_+, \hat{\mathcal{N}}_-) b_\pm^\dagger + g_\pm(\hat{N}, \hat{\mathcal{N}}_+, \hat{\mathcal{N}}_-) B^\dagger b_\mp . \tag{9}$$

Here we assert that the coefficient functions depend on the boson number $\hat{N}=B^\dagger B$ and the quasiboson number operators $\hat{\mathcal{N}}_\pm=b^\dagger_\pm b_\pm$. Using symmetry we can show that $f_+(N,\mathcal{N}_+,\mathcal{N}_-)=f_-(N,\mathcal{N}_-,\mathcal{N}_+)$, with an identical relation for $g_\pm$.

Here we make the assumption, which later will appear to be consistent, that within a space that contains only b_+ operators, the operators b_+ and $b^\dagger_+$ behave as bosons. More precisely

$$[b_\pm,b^\dagger_\pm]=1-\alpha(\hat{N},\hat{\mathcal{N}}_+,\hat{\mathcal{N}}_-)\hat{\mathcal{N}}_\mp. \tag{10}$$

Here α is symmetric under interchange of its last two arguments. Using this relation we have concluded that the easiest way to obtain explicit forms for f and g is to use the equality of the norm of a state and its mapping and we find that $f_+(0,\mathrm{i},0)=1$. Similar techniques can be used to derive

$$g^2_+(\mathrm{i},0,\mathcal{N}_-)=\frac{1}{\mathcal{N}_-+1}. \tag{11}$$

Using this relation, one can also derive that

$$f^2_+(N,\mathcal{N}_+,0)=\frac{(N+\mathcal{N}_+)}{\mathcal{N}_+}. \tag{12}$$

Some of the missing information can now be derived from the commutation relations of individual boson operators. Let us first study $[\Gamma(a^\dagger_+),\,\Gamma(a_+)]=-1$. For $\mathcal{N}_-=0$ we find

$$-1+g^2_+(N,\mathcal{N}_+,0)N(1-\alpha\mathcal{N}_+)=-1. \tag{13}$$

This has two solutions, $g^2_+(N,\mathcal{N}_+,0)=0$, or $\alpha|_{\mathcal{N}_-=0}=1/\mathcal{N}_+$. We shall argue that the last solution is the right one. Similarly $\mathcal{N}_+=0$ leads to the equation

$$-f^2_+(N,1,\mathcal{N}_-)(1-\alpha\mathcal{N}_-)-1=-1. \tag{14}$$

Again, $f^2_+(N,1,\mathcal{N}_-)=0$, or $\alpha|_{\mathcal{N}_+=0}=1/\mathcal{N}_-$.

The same solution can be found from $[\Gamma(a^\dagger_+),\,\Gamma(a^\dagger_-)]=0$. Finally, the relation $[\Gamma(a^\dagger_+),\,\Gamma(a_-)]$ can be used to show that all matrix elements of $[b^\dagger_+,b_-]$ are zero, except when we take the expectation value

$$(N,1,1|[b^\dagger_+,b_-]|N,1,-1). \tag{15}$$

This can be explicitly evaluated to be 1, so that we can capture the whole commutation relation in the form

$$[b_-,b^\dagger_+]=b^\dagger_+\frac{1}{\mathcal{N}_++\mathcal{N}_-+1}b_-. \tag{16}$$

Finally what about the product $\Gamma(a^\dagger_+)\Gamma(a^\dagger_-)$ used to define $B^\dagger$? It evaluates to

$$\begin{aligned}&g_+(N,\mathcal{N}_+,\mathcal{N}_-)f_-(N-1,\mathcal{N}_+,\mathcal{N}_-+1)B^\dagger(\mathcal{N}_-+1-\alpha\mathcal{N}_+)\\&\quad+f_+(N,\mathcal{N}_+,\mathcal{N}_-)g_-(N,\mathcal{N}_+-1,\mathcal{N}_-)B^\dagger\mathcal{N}_+.\end{aligned} \tag{17}$$

For $\mathcal{N}_+=0$ we find that this becomes $(N+\mathcal{N}_-)^{1/2}B^\dagger$, and for $\mathcal{N}_-=0$, $(N+\mathcal{N}_+)^{1/2}B^\dagger$, which agrees with equations (4) and (6).

To summarize, we have found the mapping

$$\Gamma(a^\dagger_\pm)=\left(\frac{N+\mathcal{N}_\pm}{\mathcal{N}_\pm}\right)^{1/2} b^\dagger_\pm+\frac{1}{N+\mathcal{N}_\pm}B^\dagger b_\mp$$

$$[B,B^\dagger]=1$$

$$[b_\alpha,b^\dagger_\beta]=\delta_{\alpha\beta}-b^\dagger_{\bar\alpha}\frac{1}{\mathcal{N}_++\mathcal{N}_-+1}b_{\bar\beta} \tag{18}$$

where a bar ovr an index means the opposite (i.e. $\mp=-$).

Now let us study the relation of the mapping described in this letter to Dirac quantization. In the classical limit the Hamiltonian is given by

$$H=\omega_+b^*_+b_++\omega_-b^*_-b_-+(\omega_++\omega_-)B^*B. \tag{19}$$

(here $b_\pm$ and B are complex canonical variables). We also have two constraints

$$\chi_1=b^*_\pm b_*=0 \qquad \chi_2=\chi^*_1=0. \tag{20}$$

Using the standard Dirac analysis [4], we can show that the Dirac bracket, taking into account the constraints, is given by

$$\{A,B\}_{\rm DB}=\{A,B\}-\{A,\chi_1\}\alpha\{\chi_2,B\}+\{B,\chi_1\}\alpha\{\chi_2,A\} \tag{21}$$

where curly brackets denote the standard Poisson bracket, and $\alpha=1/\{\chi_2,\chi_1\}=-\mathrm{i}/(b^*_+b_++b^*_-b_-)$. This result satisfies the standard conditions that Dirac brackets of the constraints with the Hamiltonian are zero, as well as that the Dirac bracket of two constraints vanishes.

We are interested in the Dirak bracket of the operators $b_\pm$. We find that the only non-zero ones are

$$\{b_\pm,b^*_\pm\}_{\rm DB}=\mathrm{i}\left[1-\frac{b^*_\mp b_\mp}{b^*_+b_+b^*_-b_-}\right] \tag{22}$$

$$\{b_-b^*_+\}_{\rm DB}=\mathrm{i}\left[-\frac{b^*_+b_-}{b^*_+b_++b^*_-b_-}\right]. \tag{23}$$

If we apply Dirac's prescription for quantiztion of a constrained system and define the Dirac bracket to be the classical limit of the commutator, we see that (22) and (23) agree with the algebraic (purely quantum mechanical) results derived previously, albeit with some ordering ambiguities. We have thus shown that the boson–quasiboson mapping is the natural generalization of Dirac quantization for a well defined quantum problem where we introduce a quantum mechanical redundant coordinate.

Let us conclude this letter by discussing the usefulness of the mapping. The idea to derive the mapping comes from a study of the harmonic excitations of the vibron model for rigid bent molecules [5, 6]. For this special case the Hamiltonian is usually taken to contain the absolute value of a two-body operator (the absolute value of a Hermitian operator is defined by diagonalization of the original operator, replacing the eigenvalues by their absolute values). Schematically,

$$H=H_2-a|O|. \tag{24}$$

In order to study the harmonic approximation, it is easier to first study the operator without absolute values. As discussed in [6], there are only two frequencies that depend on a, in the form $\omega_+ = 2-a$, $\omega_- = 2+a$. From these two frequencies we can build the spectrum $E_x = 2(n_+ + n_-) - a(n_+ - n_-)$. Using the absolute value of the approximate eigenvalues of O, we should just replace $-a(n_+ - n_-)$ by $-a|n_+ - n_-|$. The complicated expression obtained in this way is most easily expressed in the boson–quasifermion space as $E_x = 4N + (2-a)n$. In other words the only way to have a simple expression for the harmonic approximation to the absolute value operator is to perform the mapping discussed in this paper.

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